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Chebyshev spectral method for the Bratu problem

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Abstract

A novel and effective collocation method for solving the nonlinear Bratu differential equation is presented in this work. The application of our spectral collocation method enables us to get a system of nonlinear equations that solved using Newton's iterative approach to get the approximate solution. Furthermore, Also, numerical comparisons between the presented procedure with different numerical methods are held. The numerical outcomes confirm that the proposed procedure is precise, computationally efficient, and easy to implement.

Key Words:

Chebyshev polynomials; Collocation method; Bratu differential equation

1. Introduction:

Spectral methods are among the most widely utilized numerical approaches developed and adapted to handle numerically specific differential equations. The principle idea behind these methods is to express the unknown solution of the differential equation as a truncated series of basis functions, most likely orthogonal functions, for example, the ultraspherical polynomials, Chebyshev polynomials, Legendre polynomials, or others. The main feature of these methods fibs in their ability to reach acceptably accurate results with substantially fewer degrees of freedom. Recently, extensive research dealing with the three versions of spectral methods has appeared. For example, the authors in [Atta et al., 2019] presented a spectral method for solving the fractional initial value problems. In [Abd-Elhameed et al., 2024], a new spectral method for solving the fractional delay differential equations was established. In addition, In [Youssri and Atta, 2024a], a collocation algorithm is applied to solve the fractional integro-differential equation. Fore more studies see [Atta et al., 2021,Abd-Elhameed et al., 2023,Abdelghany et al., 2023,Atta, 2024].

Orthogonal polynomials [Alhaidari, 2024, Bertola et al., 2024, Atta, 0700, AlRawashdeh, 2024] have attracted a lot of attention from scholars in recent years because of their unique ability to describe smooth solutions as a linear combination of these functions. We discover that these polynomials clearly affect the ability to produce high-precision digital solutions using various numerical methods, including spectral methods and others, because of their orthogonal property and the properties of their roots. One of the crucial orthogonal polynomials contributed to the development of these methods is the Chebyshev polynomial. Numerous authors have examined Chebyshev polynomials from both a theoretical and practical perspective due to their importance in various mathematical fields. As an illustration, see [Youssri and Atta, 2024b,Atta et al., 2022, Tian and Xu, 2024].

In mathematical physics, the Bratu differential equation is a second-order nonlinear ordinary differential equation that results from the study of boundary value issues. Because of its exponential component, the Bratu equation is extremely nonlinear. It is frequently employed to explain a broad variety of physical phenomena, such as the spreading of flames in combustion theory and the buckling of elastic plates. Due to the hardship of obtaining its analytical solutions, there were many attempts to solve this problem; for example, the author in [Aydınlık and Kırı¸s, 2024] presented an efficient method for solving fractional integral and differential equations of Bratu type. In [Yakusak and Taiwo, 2024], the authors proposed approximate solution of Bratu differential equation using Falker-type method. In [Theophilus et al., 2024], the authors proposed K-step block hybrid nystrom-type method for the solution of Bratu problem with impedance boundary condition.

The paper is organized as follows: In Section 2, a summary of the first kind Chebyshev polynomials and their shifted ones are presented. Section 3 introduces a numerical technique for solving the nonlinear Bratu differential equation using the spectral collocation method. Section 4 provides numerical examples to illustrate the theoretical results. Lastly, Section 5 contains the conclusions.

2. An account on the first kind Chebyshev polynomials and their shifted ones

It is well known that the first kind Chebyshev polynomials are defined on $[-1,1]$ by

$$T_n(x) = \cos n\,\theta, \quad x = \cos\theta.$$

The orthogonality relation on $[-1,1]$, is

$$\int_{-1}^{1} \frac{T_m(x)\,T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n \neq 0, \\ \pi, & m = n = 0. \end{cases}$$

They may be generated by the recurrence relation

$$T_{i+1}(x) = 2\,x\,T_i(x) - T_{i-1}(x), \quad i = 1,2,3,\ldots,$$

starting from $T_0(x) = 1$ and ${}_1(x) = x$,

The modified first kind Chebyshev polynomials $\{T_i^*(x), i = 0,1,2,\ldots\}$ are a series of orthogonal polynomials on [0,1] that can be found using the recurrence formula given below:

$$T_{i+1}^*(x) = 2\,(2\,x-1)\,T_i^*(x) - T_{i-1}^*(x), \quad T_0^*(x) = 1, \quad T_1^*(x) = 2\,x-1. \tag{2}$$

$T_i^*(x)$ form a complete orthogonal system for $L^2_{\omega(x)}[0,1]$, where $\omega(x) = \frac{1}{\sqrt{x\,(1-x)}}$.

The orthogonality relation satisfied by $T_i^*(x)$ is

$$\int_0^1 T_i^*(x)\,T_j^*(x)\,\omega(x)\,d\,x = \frac{\pi}{2\,\theta_i}\,\delta_{i,j}, \tag{3}$$

where

$$\theta_i = \begin{cases} \frac{1}{2}, & if \quad i = 0, \\ 1, & if \quad i > 0, \end{cases} \tag{4}$$

and

$$\delta_{i,j} = \begin{cases} 1, & if \;\; i = j, \\ 0, & if \;\; i \neq j. \end{cases} \tag{5}$$

Moreover, the power form of $T_i^*(x)$ can be expressed as

$$T_i^*(x) = i\sum_{s=0}^{i} \frac{(-1)^{i-s}\,2^{2s}\,(i+s-1)!}{(i-s)!\,(2s)!}\,x^s, \quad i > 0. \tag{6}$$

Theorem 1 For every positive integer q, the qth derivative of $T_j^*(x)$ can be expressed in terms of their original polynomials as:

$$D^q\,T_j^*(x) = \sum_{p=0\,(j+p+q)\;even}^{j-q} B_{j,p,q}\;T_p^*(x), \tag{7}$$

where

$$B_{p,j,q} = \frac{j\,2^{2\,q}\,\theta_p\,(q)_{\frac{1}{2}(j-p-q)}}{\left(\frac{1}{2}(j-p-q)\right)!\left(\frac{1}{2}(j+p+q)\right)_{1-q}}, \tag{8}$$

where θ_p defined in (4).

Proof. The proof of this theorem can be found in [Abd–Elhameed et al., 2022].

Corollary 1 The first derivative of $T_i^*(x)$ can be expressed as

$$\frac{d\,T_i^*(x)}{d\,x} = \sum_{p=0}^{i-1} \lambda_{p,i}\,T_p^*(x), \quad i \geq 1, \tag{9}$$

where

$$\lambda_{p,i} = 2\,i \begin{cases} 1, & (i+p+1)\ \text{even},\ p = 0, \\ 2, & (i+p+2)\ \text{even}, \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

Corollary 2 The second derivative of $T_i^*(x)$ can be expressed as

$$\frac{d^2\,T_i^*(x)}{d\,x^2} = \sum_{p=0}^{i-2} \beta_{p,i}\,T_p^*(x), \quad i \geq 2, \tag{11}$$

where

$$\beta_{p,i} = 2\,i\,(i-p)\,(i+p) \times \begin{cases} 1, & (i+p+2) \text{ even}, \ p=0, \\ 2, & (i+p+2) \text{ even}, \\ 0, & \text{otherwise}. \end{cases} \quad (12)$$

Proof. The proof of Corollaries 1 and 2 can be easily obtained after putting $q = 1, 2$ respectively and simplifying the result in Theorem 1.

3 Collocation approach for the nonlinear Bratu differential equation

Consider the following nonlinear Bratu differential equation [ODETUNDE et al., 2023,Raja and Ahmad, 2014]:

$$\xi'' + \lambda\, e^{\xi} = 0; \quad 0 \le x \le 1, \quad (13)$$

subject to the initial conditions:

$$\xi(0) = a_1, \quad \xi'(0) = a_2, \quad (14)$$

or the boundary conditions:

$$\xi(0) = a_3, \quad \xi(1) = a_4, \quad (15)$$

where λ, is a given real parameter, and a_1, a_2 a_3 and a_4 are constants.

The collection of $T_i^*(x)$ form an orthogonal basis of $L^2_{\omega(x)}(0,1)$. This means that for any given function $\xi(t) \in L^2_{\omega(x)}(0,1)$, one has

$$\xi(x) = \sum_{i=0}^{\infty} c_i\, T_i^*(x), \quad (16)$$

and approximated as

$$\xi(x) \approx \xi_N(x) = \sum_{i=0}^{N} c_i\, T_i^*(x). \quad (17)$$

By virtue of Corollary 1 along with the expansion (17), the residual $\mathrm{R}(x)$ of Eq. (13) is given by

$$\begin{aligned} \mathrm{R}(x) &= \xi_N'' + \lambda\, e^{\xi_N} \\ &= \sum_{i=0}^{N} c_i\, \frac{d^2\, T_i^*(x)}{d\, x^2} + \lambda\, e^{\sum_{i=0}^{N} c_i\, T_i^*(x)} \\ &= \sum_{i=0}^{N} \sum_{p=0}^{i-2} c_i\, \beta_{p,i}\, T_p^*(x) + \lambda\, e^{\sum_{i=0}^{N} c_i\, T_i^*(x)}, \end{aligned} \quad (18)$$

Moreover, we get the following initial conditions

$$\begin{aligned} &\sum_{i=0}^{N} c_i\, T_i^*(0) = a_1, \\ &\sum_{i=0}^{N} c_i\, \frac{d\, T_i^*(0)}{d\, x} = a_2, \end{aligned} \quad (19)$$

or the boundary conditions

$$\begin{aligned} &\sum_{i=0}^{N} c_i\, T_i^*(0) = a_3, \\ &\sum_{i=0}^{N} c_i\, T_i^*(1) = a_4, \end{aligned} \quad (20)$$

Now, the application of collocation method enables us to get the following $(N+1)$ algebraic system of equations in the unknown expansion coefficients c_i

$$\begin{aligned} &\mathrm{R}(x_i) = 0, \quad i = 1, 2, \ldots, N-1, \\ &\sum_{i=0}^{N} c_i\, (-1)^i = a_1, \\ &\sum_{i=0}^{N} \sum_{p=0}^{i-1} c_i\, \lambda_{p,i}\, (-1)^p = a_2, \end{aligned} \quad (21)$$

or

$$\begin{aligned} &\mathrm{R}(x_i) = 0, \quad i = 1, 2, \ldots, N-1, \\ &\sum_{i=0}^{N} c_i\, (-1)^i = a_3, \\ &\sum_{i=0}^{N} c_i = a_4, \end{aligned} \quad (22)$$

where $\{x_i : i = 1, 2, \ldots, N-1\}$ are the first $(N-1)$ distinct roots of $T_i^*(x)$.

And hence, Eq. (21) or (22) can be solved with the aid of the well-known Newtonâ€™s iterative method.

4 Illustrative examples

Test Problem 1 [ODETUNDE et al., 2023]

Consider the following equation

$$\xi'' - 2\, e^{\xi} = 0; \quad 0 \le x \le 1, \quad (23)$$

subject to the initial conditions:

$$\xi(0) = \xi'(0) = 0, \quad (24)$$

where the exact solution is

$$\xi(x) = -2\ln(\cos(x)).$$

Table 1 shows the maximum absolute errors at different values of N. Table 2 presents a comparison of absolute errors between our method and method in [ODETUNDE et al., 2023]. Figure 1 illustrates the absolute errors at different values of N. These results prove that the approximate solution is quite near to the precise one.

Table 1: Maximum absolute errors of Example 1 .

N	5	10	15	20	25	30
Error	1.5868×10^{-3}	9.0980×10^{-7}	4.4376×10^{-9}	2.7911×10^{-13}	6.3060×10^{-14}	6.6613×10^{-16}

Table 2: Comparison of absolute errors for Example 1.

x	Our method	Method in [ODETUNDE et al., 2023]
0.1	3.46945×10^{-17}	1.4×10^{-10}
0.2	4.16334×10^{-17}	3.499×10^{-8}
0.3	1.52656×10^{-16}	9.1185×10^{-7}
0.4	2.77556×10^{-17}	9.3271×10^{-6}
0.5	2.77556×10^{-16}	5.73698×10^{-5}
0.6	5.55112×10^{-17}	2.567388×10^{-4}
0.7	4.44089×10^{-16}	9.260041×10^{-4}
0.8	2.22045×10^{-16}	2.8639825×10^{-3}
0.9	2.22045×10^{-16}	7.9152872×10^{-3}
1	6.66134×10^{-16}	2.014183×10^{-3}

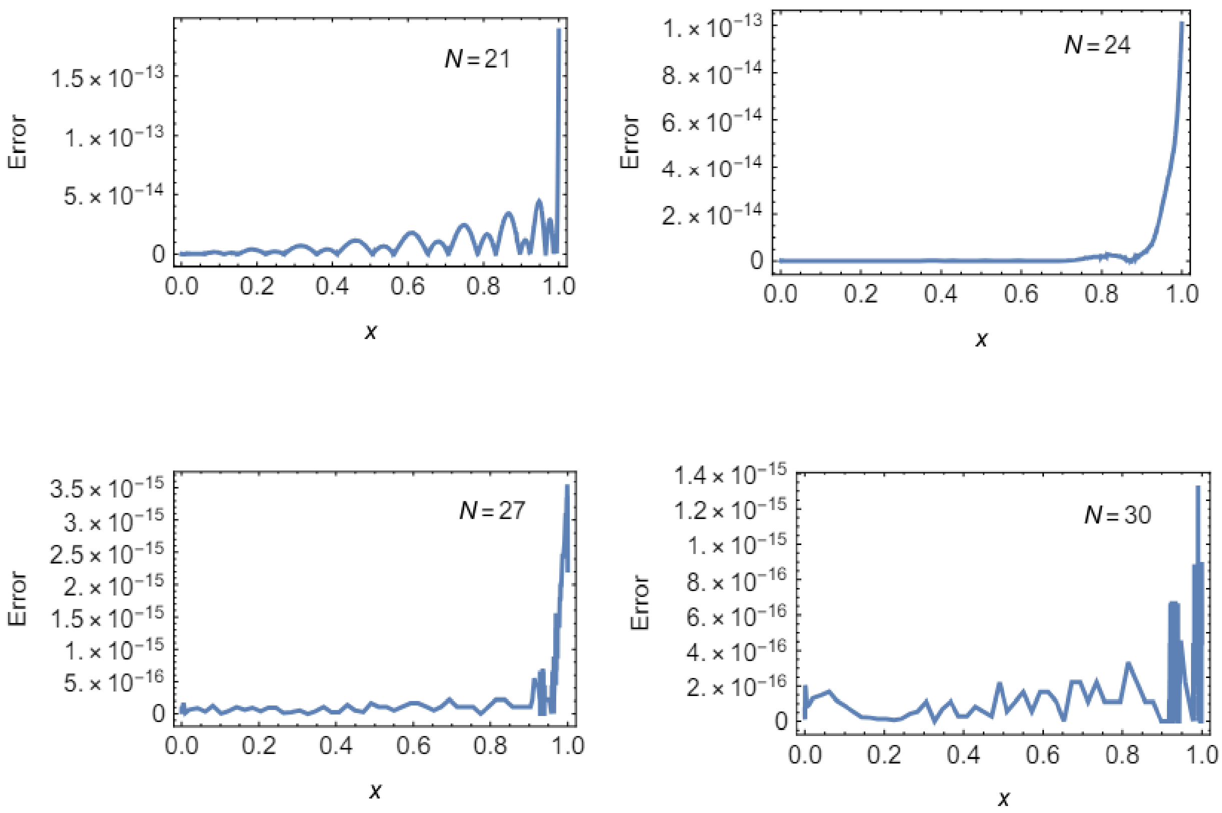

Figure 1: The absolute errors of Example 1 at different values of N

Test Problem 2 [ODETUNDE et al., 2023]

Consider the following equation

$$\xi'' - \pi^2 e^{\xi} = 0; \quad 0 \le x \le 1, \tag{25}$$

subject to the initial conditions:

$$\xi(0) = \xi'(0) = 0, \tag{26}$$

where the exact solution is

$$\xi(x) = -\log(1 - \cos(\pi(x + 0.5))).$$

Table 3 shows the maximum absolute errors at different values of N. Figure 2 illustrates the absolute errors at different values of N. Figure 3 shows the absolute errors (left) at $N = 30$ and approximate solution (right). These results prove that the approximate solution is quite near to the precise one.

Table 3: Maximum absolute errors of Example 2 .

N	5	10	15	20	25	30
Error	8.072×10^{-3}	3.326×10^{-6}	3.980×10^{-9}	2.775×10^{-12}	1.216×10^{-12}	5.689×10^{-15}

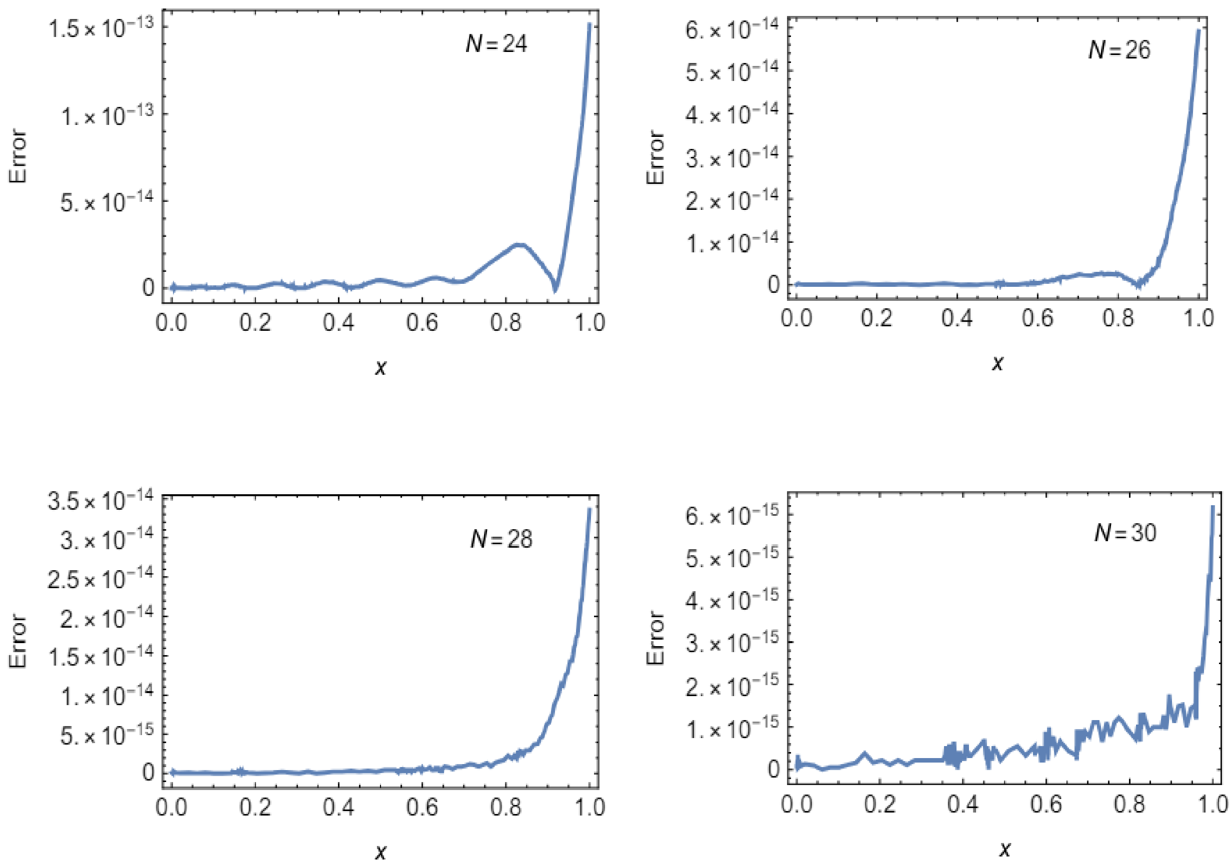

Figure 2: The absolute errors of Example 2 at different values of N

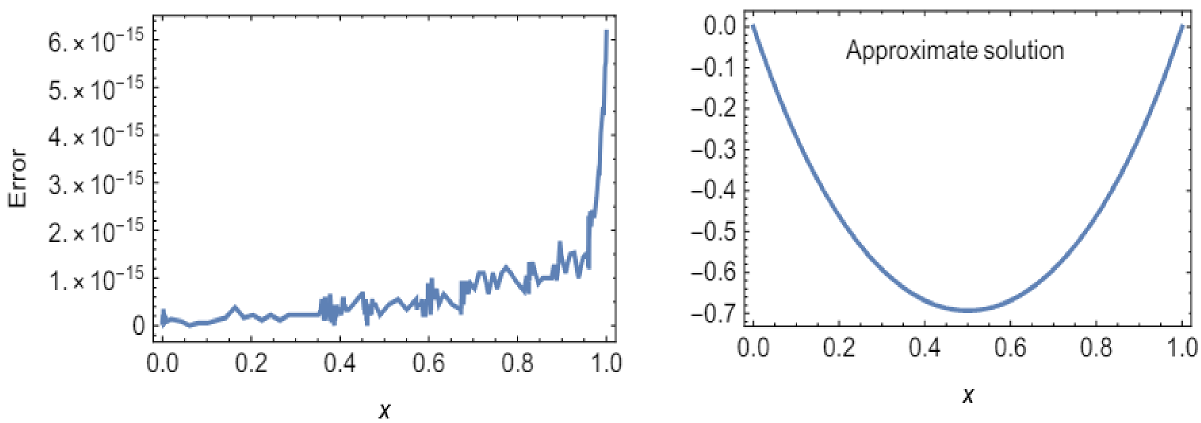

Figure 3: The absolute errors (left) and approximate solution (right) of Example 2

Test Problem 3 [Raja and Ahmad, 2014]

Consider the following equation

$$\xi'' + e^{\xi} = 0; \quad 0 \le x \le 1, \tag{27}$$

subject to the boundary conditions:

$$\xi(0) = \xi(1) = 0, \tag{28}$$

where the exact solution is $\xi(x) = -2\log\left(\frac{\cosh\left(\frac{\theta}{2}\left(x-\frac{1}{2}\right)\right)}{\cosh\left(\frac{\theta}{4}\right)}\right)$ at $\theta = 2.3576$.

Table 4 shows the maximum absolute errors at different values of N. Figure 4 illustrates the absolute errors at different values of N. Table 5 presents a comparison of absolute errors between our method and method in [Raja and Ahmad, 2014]. These results prove that the approximate solution is quite near to the precise one.

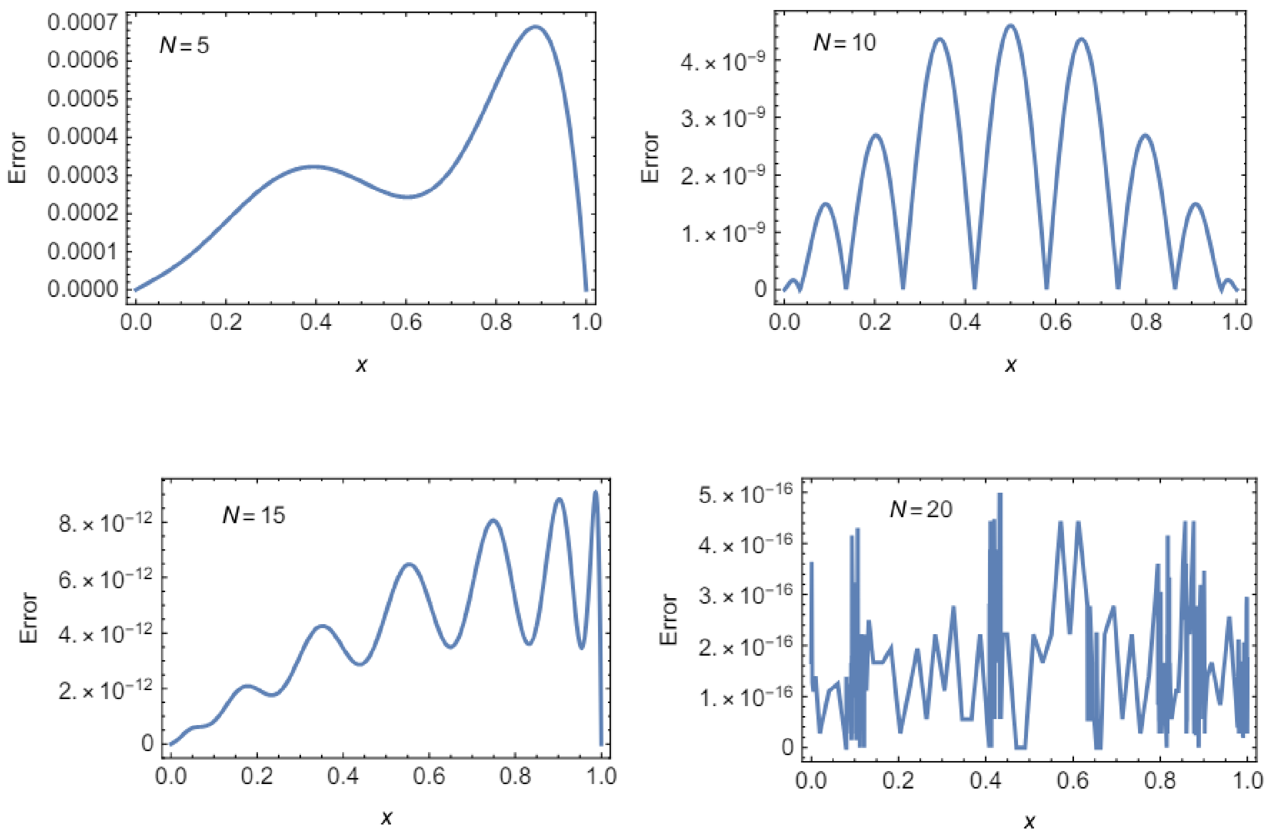

Figure 4: The absolute errors of Example 3 at different values of N

Table 4: Maximum absolute errors of Example 3 .

N	4	8	12	16	20
Error	7.43835×10^{-4}	2.03968×10^{-7}	1.29606×10^{-10}	1.00087×10^{-13}	3.88578×10^{-16}

Table 5: Comparison of absolute errors for Example 3.

x	Our method	Method in [Raja and Ahmad, 2014]
0.1	1.52656×10^{-16}	6.2868×10^{-10}
0.3	1.11022×10^{-16}	1.2860×10^{-8}
0.5	1.66533×10^{-16}	3.4980×10^{-10}
0.7	1.11022×10^{-16}	1.7153×10^{-9}
0.9	1.52656×10^{-16}	6.9360×10^{-10}

5 Concluding remarks

A numerical study for the Bratu differential equation is introduced using he shifted Chebyshev polynomials of the first kind as basis functions. The suggested method rely on the principle of reducing the problem to nonlinear algebraic equations that can be solved using appropriate numerical techniques. Some numerical examples have been presented that powerfully illustrate the accuracy of the presented study to the proposed Bratu differential equation.

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